MTH 534 equation sheet

13 Algebra of differential forms

Wedge product (exterior product)

$\wedge : \bigwedge^p \times \bigwedge^q \mapsto \bigwedge^{p+q}$

$\alpha \wedge \alpha = 0$, $\beta \wedge \alpha = -\alpha \wedge \beta \in \bigwedge^2$ for $\alpha, \beta \in \bigwedge^1$

Distributive (bilinear) & associative:

$\alpha, \beta, \gamma \in \bigwedge^1$, f fn

$(\alpha + \beta) \wedge \gamma = \alpha \wedge \gamma + \beta \wedge \gamma$

$f(\alpha \wedge \gamma) = (f\alpha) \wedge \gamma$

$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$

Differential forms

Space of p-forms:

$\bigwedge^p = \langle \{dx^{i_1} \wedge \cdots \wedge dx^{i_p}\} \rangle$, $1 \le i_1 < \cdots < i_p \le n$

\# indep p-forms: $\dim \bigwedge^p = \binom{n}{p} = \frac{n!}{p!(n-p)!}$

Linear map on 1-forms: $A : \bigwedge^1 \mapsto \bigwedge^1$, $\alpha \mapsto A(\alpha)$, $A(f\alpha + \beta) = fA(\alpha) + A(\beta)$

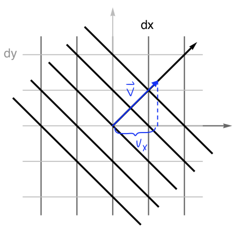
Linear map A on p-forms: $A : \bigwedge^p \mapsto \bigwedge^p$

$\alpha^1 \wedge \cdots \wedge \alpha^p \mapsto A\alpha^1 \wedge \cdots \wedge A\alpha^p$,

$A(\omega) = |A|\omega = (\det A)\omega$ for all $\omega \in \bigwedge^n(V)$

Products: $\alpha \in \bigwedge^p$, $\beta \in \bigwedge^q$, $\beta \wedge \alpha = (-1)^{pq} \alpha \wedge \beta$

p-form $\beta \in \bigwedge^p(\mathbb{R}^n)$ decomposable if $\exists \alpha_i \in \bigwedge^1(\mathbb{R}^n)$ s.t. $\beta = \alpha_1 \wedge \cdots \wedge \alpha_p$. All elements of $\bigwedge^{n-1}(\mathbb{R}^n)$ are decomposable. γ decomposable $\Rightarrow \gamma \wedge \gamma = 0$ but converse false.



14 Hodge duality

Bases for differential forms

M is n-dim surface w/ coordinates (x^i)

Coordinate basis of 1-forms: $\{dx^i\}$

Arbitrary basis: if $\{\sigma^i\}$ is a basis of $\bigwedge^1$, then $\{\sigma^I\} = \{\sigma^{i_1} \wedge \cdots \wedge \sigma^{i_p}\}$ is a basis for $\bigwedge^p$ where index set $I = \{i_1, \ldots, i_p\}$, $1 \le i_1 < \cdots < i_p \le n$, $p \le n$

Metric tensor, inner product

Def inner product on 'vector space' V as map $g : V \times V \mapsto \mathbb{R}$ s.t.

1) (bi)linear: $g(f\alpha + \beta, \gamma) = fg(\alpha, \gamma) + g(\beta, \gamma)$

2) symmetric: $g(\beta, \alpha) = g(\alpha, \beta)$

3) non-degenerate: $g(\alpha, \beta) = 0 \; \forall \; \beta \Rightarrow \alpha = 0$

For inner product g (metric tensor) on $\bigwedge^1$:

Components in basis $\{\sigma^i\}$: $g^{ij} = g(\sigma^i, \sigma^j)$.

Can assume wlog that basis is orthonorm, $g(\sigma^i, \sigma^j) = \pm\delta^{ij}$, $g(\sigma^I, \sigma^J) = \pm\delta^{IJ}$, $g(1,1) = 1$.

Line element $ds^2 = g_{ij} dx^i dx^j$ in orthonorm basis.

Signature of metric $= \#$ minus signs.

	flat	curved
$s = 0$	Euclidean	Riemannian
$s = 1$	Minkowskian	Lorentzian

Action of metric g on 2-forms:

$g(\alpha \wedge \beta, \gamma \wedge \delta) = \begin{vmatrix} g(\alpha, \gamma) & g(\alpha, \delta) \\ g(\beta, \gamma) & g(\beta, \delta) \end{vmatrix}$

Schwarz inequality: $g(\alpha, \beta)^2 \le g(\alpha, \alpha) g(\beta, \beta)$ in $\mathbb{E}^n$

Reverse Schwarz: $g(\alpha, \beta)^2 \ge g(\alpha, \alpha) g(\beta, \beta)$ in $\mathbb{M}^2$

Orientation

$\{\sigma^i\}$ is orthonorm basis of $\bigwedge^1(M)$.

Orientation $\omega = \sigma^1 \wedge \cdots \wedge \sigma^n$ on M chooses one of two unit n-forms (other is $-\omega$). $g(\omega, \omega) = (-1)^s$

Hodge dual

$* : \bigwedge^p \mapsto \bigwedge^{n-p}$ is a linear map defined by: $=$

$\alpha \wedge *\beta = g(\alpha, \beta)\omega \in \bigwedge^n$ $(= \beta \wedge *\alpha)$

In any sig/dim: $*1 = \omega$, $*\omega = (-1)^s$,

$g(\alpha, \beta) = (-1)^s * (\alpha \wedge *\beta)$

General case: $\omega = \sigma^1 \wedge \cdots \wedge \sigma^n$, $*\sigma^I = g(\sigma^I, \sigma^I)\sigma^J$ where $\sigma^I = \sigma^1 \wedge \cdots \wedge \sigma^p$, $\sigma^J = \sigma^{p+1} \wedge \cdots \wedge \sigma^n$

$** = (-1)^{p(n-p)+s}$ always. In $s = 0$, $n = 3$: $** = 1$

Dot and cross product

In any dim/sig: $g(\alpha, \beta) = \alpha \cdot \beta = (-1)^s * (\alpha \wedge *\beta)$

In 3D, $\alpha, \beta \in \bigwedge^1$: $\alpha \times \beta = *(\alpha \wedge \beta)$

15 Differentiation of differential forms

Exterior diff.: $\alpha = f dx^I = f dx^{i_1} \wedge \cdots \wedge dx^{i_p}$,

$d\alpha = df \wedge dx^I$ in coord basis.

Thm: $\exists!\; d : \bigwedge^p \mapsto \bigwedge^{p+1}$ satisfying properties:

1) linear: $d(a\alpha + \beta) = a d\alpha + d\beta$, $a =$ const

2) product rule: $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$ if $\alpha = p$-form

3) $d^2\alpha = 0 \Leftrightarrow d(f d\alpha) = df \wedge d\alpha$

4) $df = \frac{\partial f}{\partial x^i} dx^i$ $(= d(f))$

$$\underbrace{\bigwedge^0 \xrightarrow[\text{grad}]{d} \underbrace{\bigwedge^1 \xrightarrow[\text{curl}]{d} \bigwedge^2}_{*} \xrightarrow[\text{div}]{d} \bigwedge^3}_{*}$$

For 0-form f, 1-form $F = \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$:

$\text{grad}(f) = df = \vec{\nabla} f \cdot d\vec{\mathbf{r}}$ $(= \frac{\partial f}{\partial x^1} dx^1 + \ldots)$

$\text{curl}(F) = *dF = (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot d\vec{\mathbf{r}}$

$\text{div}(F) = *d * F = \vec{\nabla} \cdot \vec{\mathbf{F}}$

Laplacian $\triangle f = *d * df$

Maxwell's equations

$\vec{\nabla} \cdot \vec{\mathbf{E}} = 4\pi\rho$	$d * E = 4\pi * \rho$
$\vec{\nabla} \cdot \vec{\mathbf{B}} = 0$	$d * B = 0$
$\vec{\nabla} \times \vec{\mathbf{E}} + \dot{\vec{\mathbf{B}}} = 0$	$dE + *\dot{B} = 0$
$\vec{\nabla} \times \vec{\mathbf{B}} - \dot{\vec{\mathbf{E}}} = 4\pi\vec{\mathbf{J}}$	$dB - *\dot{E} = 4\pi * J$
$\vec{\nabla} \cdot \vec{\mathbf{J}} + \dot{\rho} = 0$	$*d * J + \dot{\rho} = 0$
Ansatz:	
$\vec{\mathbf{B}} = \vec{\nabla} \times \vec{\mathbf{A}}$	$B = *dA$
$\vec{\mathbf{E}} = -\vec{\nabla}\mathbf{\Phi} - \dot{\vec{\mathbf{A}}}$	$E = -d\Phi - \dot{A}$

In $\mathbb{M}^4(*, d)$, $\omega = \bar{\omega} \wedge dt$ $[\mathbb{R}^3(\bar{*}, \bar{d})$, $\bar{\omega} = dx \wedge dy \wedge dz]$:

$df = \bar{d}f + \dot{f}dt$

$d\alpha = \bar{d}\alpha + dt \wedge \dot{\alpha} \Longrightarrow$

$*\bar{E} = (\bar{*}\bar{E}) \wedge dt$

$*f dt = \bar{*}f$

$F = \bar{E} \wedge dt + \bar{*}\bar{B}$, via ansatz $A = \bar{A} - \Phi dt$ $(J = \bar{J} - \rho dt)$, reduces to $F = dA$, $d * F = 4\pi * J$

Orthogonal coordinates

(u, v, w) ortho coords in $\mathbb{E}^3$, $\{\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}\}$ orthonorm

$d\vec{\mathbf{r}} = h_u du \hat{\mathbf{u}} + h_v dv \hat{\mathbf{v}} + h_w dw \hat{\mathbf{w}}$

$ds^2 = d\vec{\mathbf{r}} \cdot d\vec{\mathbf{r}} = h_u^2 du^2 + h_v^2 dv^2 + h_w^2 dw^2$

$\{h_u du, h_v dv, h_w dw\} =$ orthonorm basis of 1-forms

$d\vec{\mathbf{r}} = \frac{\partial \vec{\mathbf{r}}}{\partial u} du + \frac{\partial \vec{\mathbf{r}}}{\partial v} dv + \frac{\partial \vec{\mathbf{r}}}{\partial w} dw$

$h_u = \left| \frac{\partial \vec{\mathbf{r}}}{\partial u} \right|$, $\hat{\mathbf{u}} = \frac{1}{h_u} \frac{\partial \vec{\mathbf{r}}}{\partial u}$

$df = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv + \frac{\partial f}{\partial w} dw = \vec{\nabla} f \cdot d\vec{\mathbf{r}}$

$\vec{\nabla} f = \frac{1}{h_u} \frac{\partial f}{\partial u} \hat{\mathbf{u}} + \frac{1}{h_v} \frac{\partial f}{\partial v} \hat{\mathbf{v}} + \frac{1}{h_w} \frac{\partial f}{\partial w} \hat{\mathbf{w}}$

$\vec{\nabla} \times \vec{\mathbf{F}} = \frac{1}{h_u h_v h_w} \begin{vmatrix} h_u \hat{\mathbf{u}} & h_v \hat{\mathbf{v}} & h_w \hat{\mathbf{w}} \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_u F^u & h_v F^v & h_w F^w \end{vmatrix}$

$\vec{\nabla} \cdot \vec{\mathbf{F}} = \frac{1}{h_u h_v h_w} \left(\hat{\mathbf{u}} \frac{\partial}{\partial u} + \hat{\mathbf{v}} \frac{\partial}{\partial v} + \hat{\mathbf{w}} \frac{\partial}{\partial w} \right) \cdot (h_v h_w F^u \hat{\mathbf{u}} + h_w h_u F^v \hat{\mathbf{v}} + h_u h_v F^w \hat{\mathbf{w}})$

16 Integration of differential forms

Formalizing vectors $\Leftrightarrow$ 1-forms

$\vec{\mathbf{F}}$ vector field has corresponding 1-form $F = \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$. Map has inverse $g(F, d\vec{\mathbf{r}}) = \vec{\mathbf{F}}$ which preserves inner product, $g(\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}, \vec{\mathbf{G}} \cdot d\vec{\mathbf{r}}) = \vec{\mathbf{F}} \cdot \vec{\mathbf{G}}$. If $d\vec{\mathbf{r}}$ in orthonorm basis of vectors $= \sigma^i \hat{\mathbf{e}}_i$ for some 1-forms σ^i, then $\sigma^i = \pm \hat{\mathbf{e}}_i \cdot d\vec{\mathbf{r}}$ and $g(\sigma^i, \sigma^j) = \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \pm\delta_{ij}$ s.t. $\{\sigma^i\}$ is orthonorm basis of 1-forms.

Line, surface, volume integrals

Line: $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C F$, $F = \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$

Surface: $\int_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{A}} = \int_S *F$, $*F = \vec{\mathbf{F}} \cdot d\vec{\mathbf{A}}$

Volume: $*f = f dV$, define $\int_p f = f(p)$ for 0-form

Two integrands equal after being integrated over *any* domain $\Rightarrow$ integrands themselves equal.

Vector calculus theorems

Recall single-var FTC: $\int_a^b \frac{df}{dx} dx = f(b) - f(a)$

Gradient, Stokes, divergence theorems

$\int_A^B \vec{\nabla} f \cdot d\vec{\mathbf{r}} = f\big\|_A^B$	$\int_C df = \int_{\partial C} f = f\big\|_A^B$
$\int_S \vec{\nabla} \times \vec{\mathbf{F}} \cdot d\vec{\mathbf{A}} = \oint_{\partial S} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$	$\int_S dF = \int_{\partial S} F$
$\int_R \vec{\nabla} \cdot \vec{\mathbf{F}} dV = \oint_{\partial R} \vec{\mathbf{F}} \cdot d\vec{\mathbf{A}}$	$\int_R d * F = \int_{\partial R} *F$

Most general Stokes' theorem: $\int_R d\alpha = \int_{\partial R} \alpha$

for any $\alpha \in \bigwedge^p$ and any $(p+1)$-dim region R.

Integration by parts

$\int_R d\alpha \wedge \beta = \int_R \alpha \wedge \beta - (-1)^p \int_R \alpha \wedge d\beta$

(p-form α, q-form β)

Special case retrieves $\int_C f dg = \int_{\partial C} gf - \int_C g df$

17 Connections

Vector-valued differential forms

Vector-valued p-form: $\alpha^i \hat{\mathbf{e}}_i$, $\alpha^i = p$-form, $\{\hat{\mathbf{e}}_i\} =$ vector basis (chosen orthonorm)

Arbitrary basis: $\vec{\mathbf{F}} = F^i \hat{\mathbf{e}}_i$, $d\vec{\mathbf{F}} = dF^i \hat{\mathbf{e}}_i + F^i d\hat{\mathbf{e}}_i$

Exterior differentiation of vector fields

Some (not all) required properties:

($\alpha \in \bigwedge^p$, $\vec{\mathbf{v}}$, $\vec{\mathbf{w}}$ VFs (vector-valued 0-form), a const)

1) linear: $d(a\vec{\mathbf{v}} + \vec{\mathbf{w}}) = a d\vec{\mathbf{v}} + d\vec{\mathbf{w}}$

2) product rule: $d(\alpha\vec{\mathbf{v}}) = d\alpha\vec{\mathbf{v}} + (-1)^p \alpha \wedge d\vec{\mathbf{v}}$,

$d(\vec{\mathbf{v}}\alpha) = d\vec{\mathbf{v}} \wedge \alpha + \vec{\mathbf{v}} d\alpha$

Connections

Action of d on basis $\{\hat{\mathbf{e}}_i\}$: $d\hat{\mathbf{e}}_j = \omega^i{}_j \hat{\mathbf{e}}_i$

$\omega^i{}_j \equiv$ connection 1-forms, $\omega_{ij} = \Gamma_{ijk}\sigma^k$

Components of the metric tensor: $g_{ij} = \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j$, diagonal in orthonorm w/ entries ± 1 and I in $\mathbb{E}^n$

$(g_{ij})^{-1} = g^{ij} = g(\sigma^i, \sigma^j)$

$ds^2 = d\vec{\mathbf{r}} \cdot d\vec{\mathbf{r}} = (\sigma^i \hat{\mathbf{e}}_i) \cdot (\sigma^j \hat{\mathbf{e}}_j) = g_{ij}\sigma^i\sigma^j$

$ds^2 = g_{ij} dx^i dx^j$ in coordinate basis

$\omega_{ij} = \hat{\mathbf{e}}_i \cdot d\hat{\mathbf{e}}_j = g_{ik}\omega^k{}_j$

The Levi-Civita connection

A connection is Levi-Civita iff:

1) metric compatible $\omega_{ji} + \omega_{ij} = 0$

$(d(\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}) = d\vec{\mathbf{v}} \cdot \vec{\mathbf{w}} + \vec{\mathbf{v}} \cdot d\vec{\mathbf{w}})$

2) torsion-free: $d\sigma^i + \omega^i{}_j \wedge \sigma^j = 0$ $(d^2\vec{\mathbf{r}} = d(d\vec{\mathbf{r}}) = 0)$

Claim: given $d\vec{\mathbf{r}}$, $\exists!$ a Levi-Civita connection

18 Curvature

Curvature of a curve

Curvature: $\kappa = 1/r$, $r =$ radius of best-fit circle

Unit tangent: $\hat{\mathbf{T}} = d\vec{\mathbf{r}}/|d\vec{\mathbf{r}}| = d\vec{\mathbf{r}}/ds$, $s =$ arclength

Circle: $d\hat{\mathbf{T}}/ds = (1/r)\hat{\mathbf{N}} = \kappa\hat{\mathbf{N}}$, $\hat{\mathbf{N}}$ inward-pointing

$\Rightarrow \kappa ds = d\hat{\mathbf{T}} \cdot \hat{\mathbf{N}} = -d\hat{\mathbf{N}} \cdot \hat{\mathbf{T}}$, since $\hat{\mathbf{N}} \perp \hat{\mathbf{T}}$

Curvature of surfaces

Given $\mathbb{E}^3$, coords (x^1, x^2, x^3):

on surface $x^3 =$ const, $\sigma^3 = 0$.

Orthonorm basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{n}} = \hat{\mathbf{e}}_3\}$:

$\kappa ds = -d\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_i = \hat{\mathbf{n}} \cdot d\hat{\mathbf{e}}_i = +\omega_{3i}$.

Structure eqns: $\cancel{d\sigma^3} + \omega^3{}_i \wedge \sigma^i = 0$

$\Rightarrow \omega^3{}_1 \wedge \sigma^1 + \omega^3{}_2 \wedge \sigma^2 = 0$

$\omega^3{}_i = \Gamma^3{}_{ij}\sigma^j \Rightarrow$ structure eqn forces $\Gamma^3{}_{12} = \Gamma^3{}_{21}$

Shape operator $\equiv S = -(\Gamma^3{}_{ij})$, $i, j = 1, 2$ (symm.).

Eigenvalues are principal curvatures κ_1, κ_2

Mean curvature $= \text{Tr}(S) = \kappa_1 + \kappa_2$

Gaussian curvature $K = \det(S) = \kappa_1\kappa_2$

Curvature

Structure eqns: 1) $d^2\vec{\mathbf{r}} = \Theta^i \hat{\mathbf{e}}_i$, 2) $d^2\hat{\mathbf{e}}_j = \Omega^i{}_j \hat{\mathbf{e}}_i$

1) $\Theta^i = \omega^i{}_j \wedge \sigma^j + d\sigma^i$, $\Theta^i \equiv$ torsion 2-forms

$\Rightarrow$ Assume $d^2\vec{\mathbf{r}} = 0$, reduces to $0 = d\sigma^i + \omega^i{}_j \wedge \sigma^j$

2) $\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j$, $\Omega^i{}_j \equiv$ curvature 2-forms

In general $\Omega_{ij} = -\Omega_{ji}$,

$\exists \binom{n}{2} = \frac{1}{2}n(n-1)$ independent $\Omega^i{}_j$'s

Curvature in 3D

In $\mathbb{E}^3$, $0 = \Omega^i{}_j$.

Theorema Egregium:

For 2D surface, $\Omega^1{}_2 = d\omega^1{}_2 = K\omega = K\sigma^1 \wedge \sigma^2$

Interpretation: Gaussian curvature, prev def extrinsically, is an *intrisic*, purely geometric, property of the surface.

Components

Components of $\omega^i{}_j = \Gamma^i{}_{jk}\sigma^k \rightarrow$ Christoffels $\Gamma^i{}_{jk}$

$\Omega^i{}_j = \frac{1}{2} R^i{}_{jkl} \sigma^k \wedge \sigma^l$

$R^i{}_{jkl} \equiv$ components of Riemann curvature tensor

Assume $R^i{}_{jkl} = -R^i{}_{jkl}$ by convention.

Ricci curvature tensor $\equiv$ trace of Riemann tensor: $\Omega^k{}_i(\hat{\mathbf{e}}_k) = R^m{}_{imj}\sigma^j = R_{ij}\sigma^j$, Ricci components $R_{ij} \iff g(\Omega^k{}_i, \sigma_k \wedge \sigma_j) = R_{ij}$

Ricci curvature scalar $\equiv$ trace of Ricci tensor:

$R = R_{ij}g^{ij} = g(\sigma^i, R_{ij}\sigma^j)$

Bianchi identities

1) $0 = -d^2\sigma^i = \Omega^i{}_j \wedge \sigma^j$

In components: $R^i{}_{jkl} + R^i{}_{klj} + R^i{}_{ljk} = 0$

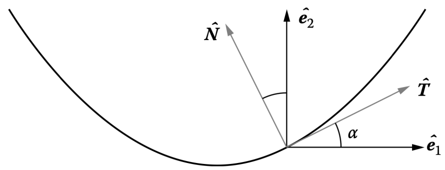
2) ("the") $0 = d^2\omega^i{}_j = d\Omega^i{}_j + \omega^i{}_k \wedge \Omega^k{}_j - \Omega^i{}_k \wedge \omega^k{}_j$

Interp: "covariant curl" of Riemann tensor vanishes

Imply symmetries: $R_{ijlk} = -R_{ijkl}$, metric compat

$R_{jikl} = -R_{ijkl}$, $R_{ij} = R_{ji}$, $R_{ijkl} = R_{klij}$

Geodesic curvature



Choose orthonorm $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2\}$

$\hat{\mathbf{T}} = \cos\alpha\hat{\mathbf{e}}_1 + \sin\alpha\hat{\mathbf{e}}_2$

$\hat{\mathbf{N}} = -\sin\alpha\hat{\mathbf{e}}_1 + \cos\alpha\hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_3 \times \hat{\mathbf{T}}$

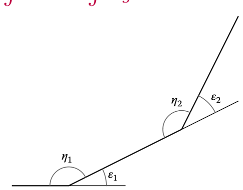
Geodesic curvature: $\kappa_g ds = d\hat{\mathbf{T}} \cdot \hat{\mathbf{N}} = d\alpha - \omega^1{}_2$

A curve with $\kappa_g = 0$ is a *geodesic*.

Geodesic triangles

Total curvature around closed curve:

$\int K\omega + \oint \kappa_g ds = 2\pi$



Piecewise smooth:

$\int K\omega + \oint \kappa_g ds = 2\pi - \sum \epsilon_i = 2\pi - \sum(\pi - \eta_i)$

Area of geodesic triangle on sphere:

$A/r^2 = 2\pi - \sum(\pi - \eta_i) = \sum \eta_i - \pi$, $i = 1\ldots 3$

Interp: A/r^2 = angle sum $-\,\pi \Rightarrow$ no similar $\triangle$'s!

In plane: $K = 0 \Rightarrow$ angle sum $= \pi$

Gauss-Bonnet theorem

Any (oriented) compact surface $\Sigma \in \mathbb{R}^3$ has a rectangular decomposition. Computing total curvature around one quadrilateral: $\int K\omega + \oint \kappa_g ds = 2\pi - \sum(\pi - \eta_i) = \sum \eta_i - 2\pi$. Summing over all quadrilaterals, κ_g terms cancel, interior angles η_i sum to 2π at each vertex, considering faces $\Rightarrow \int_\sigma K\omega = 2\pi v - 2\pi f$. Using $e - f = f$:

Gauss-Bonnet Theorem:

$\int_\Sigma K\omega = 2\pi(v - e + f) = 2\pi\chi(\Sigma)$

$\chi(\Sigma) \equiv$ Euler characteristic $= v - e + f$

is a topological invariant of the surface.

Results for a few coordinate systems

$d\vec{\mathbf{r}} = \sigma^i\hat{\mathbf{e}}_i$

$d^2\vec{\mathbf{r}} = \Theta^i\hat{\mathbf{e}}_i = \vec{\mathbf{0}}$ (torsion-free), $d^2\hat{\mathbf{e}}_j = \Omega^i{}_j\hat{\mathbf{e}}_i$

$ds^2 = d\vec{\mathbf{r}} \cdot d\vec{\mathbf{r}} = g_{ij}dx^i dx^j = g_{ij}\sigma^i\sigma^j$

Euclidean $\mathbb{R}^2$

$ds^2 = dx^2 + dy^2$, $\{dx, dy\}$, $\omega = dx \wedge dy$

$*dx = dy, *dy = -dx, *1 = dx \wedge dy, *(dx \wedge dy) = 1$

$\triangle f = *d*df = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)$

Levi-Civita: $\sigma^x = dx$, $\sigma^y = dy$

$\omega_{xx} = \omega_{yy} = \omega_{xy} = 0$

Polar coordinates in $\mathbb{R}^2$

Coord basis $\{dr, d\phi\}$, orthonorm $\{dr, rd\phi\}$,

$x = r\cos\phi$, $y = r\sin\phi$, $r^2 = x^2 + y^2$, $\tan\phi = \frac{y}{x}$

$rdr = xdx + ydy$, $r^2 d\phi = xdy - ydx$

$dx = dr\cos\phi - r\sin\phi d\phi$

$dy = dr\sin\phi + r\cos\phi d\phi$

$d\vec{\mathbf{r}} = dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} = dr\hat{\mathbf{r}} + rd\phi\hat{\boldsymbol{\phi}}$

$= \sigma^i\hat{\mathbf{e}}_i$ where $\sigma^1 = r$, $\sigma^2 = rd\phi$

$ds^2 = dx^2 + dy^2 = dr^2 + r^2d\phi^2 = d\vec{\mathbf{r}} \cdot d\vec{\mathbf{r}}$

$\omega = dr \wedge rd\phi$, $*dr = rd\phi$, $*rd\phi = -dr$, $*\omega = 1$

$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial\phi}\hat{\boldsymbol{\phi}}$, $df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial\phi}d\phi$

$\triangle f = *d*df = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\phi^2}$

Connections ($\hat{\mathbf{e}}_r = \hat{\mathbf{r}}$, $\hat{\mathbf{e}}_\phi = \hat{\phi}$):

$d\hat{\mathbf{r}} = d\phi\hat{\boldsymbol{\phi}}$

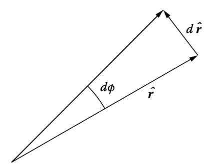
$d\hat{\boldsymbol{\phi}} = -d\phi\hat{\mathbf{r}}$

$\hat{\mathbf{r}} = \cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}$

$\hat{\boldsymbol{\phi}} = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}$

$\omega^r{}_r = 0 = \omega^\phi{}_\phi$

$\omega^r{}_\phi = -d\phi = -\omega^\phi{}_r$



Minkowski 2-space $\mathbb{M}^2$ ($s = 1$)

$d\vec{\mathbf{r}} = dt\hat{\mathbf{t}} + dx\hat{\mathbf{x}}$, $ds^2 = dx^2 - dt^2$

$\{dx, dt\}$, $\omega = dx \wedge dt$

$*dx = dt$, $*dt = dx$,

$*1 = \omega = dx \wedge dt$, $*(dx \wedge dt) = -1$

$g(dx, dx) = 1$, $g(dt, dt) = -1$

Euclidean $\mathbb{R}^3$

0-form:	f
1-form:	$F = \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$
2-form:	$\alpha = \vec{\mathbf{F}} \cdot d\vec{\mathbf{A}}$
3-form:	$\beta = fdV$

$*f = fdV$

$*F = \vec{\mathbf{F}} \cdot d\vec{\mathbf{A}}$

$f \in \bigwedge^0$, $fdV \in \bigwedge^3$,

$\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \in \bigwedge^1$, $\vec{\mathbf{F}} \cdot d\vec{\mathbf{A}} \in \bigwedge^2$

$d\vec{\mathbf{A}} = *d\vec{\mathbf{r}}$, $dV \leftrightarrow \omega$

$\vec{\mathbf{F}} \cdot \vec{\mathbf{G}} = *(F \wedge *G)$, $(\vec{\mathbf{F}} \times \vec{\mathbf{G}}) \cdot d\vec{\mathbf{r}} = *(F \wedge G)$

$\vec{\nabla} f \cdot d\vec{\mathbf{r}} = df = \nabla f$

$(\vec{\nabla} \times \vec{\mathbf{F}}) \cdot d\vec{\mathbf{r}} = *dF = \nabla \times F$

$\vec{\nabla} \cdot \vec{\mathbf{F}} = *d*F = \nabla \cdot F$

$\triangle f = \vec{\nabla} \cdot \vec{\nabla} f = *d*df = \nabla \cdot \nabla f$

Rectangular coordinates:

Vector field: $\vec{\mathbf{F}} = F_x\hat{\mathbf{x}} + F_y\hat{\mathbf{y}} + F_z\hat{\mathbf{z}}$

w/ rect. basis $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$ or $\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\}$

Displacement $d\vec{\mathbf{r}} = dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}}$

	form	basis
0	f	$\{1\}$
1	$F = F_xdx + F_ydy + F_zdz$	$\{dx, dy, dz\}$
2	$\alpha = F_xdy \wedge dz + F_ydz \wedge dx + F_zdx \wedge dy$	$\{dy \wedge dz, dz \wedge dx, dx \wedge dy\}$
3	$\beta = fdV = fdx \wedge dy \wedge dz$	$\{dx \wedge dy \wedge dz\}$

$*1 = dx \wedge dy \wedge dz$

$*dx = dy \wedge dz$

$*dy = dz \wedge dx$

$*dz = dx \wedge dy$

$*(dy \wedge dz) = dx$

$*(dz \wedge dx) = dy$

$*(dx \wedge dy) = dz$

$*(dx \wedge dy \wedge dz) = 1$

x-y plane curvature: $\hat{\mathbf{e}}_1 = \hat{\mathbf{x}}$, $\hat{\mathbf{e}}_2 = \hat{\mathbf{y}}$, $\hat{\mathbf{n}} = \hat{\mathbf{e}}_3 = \hat{\mathbf{z}}$

$d\hat{\mathbf{n}} = 0$, $S = \mathbf{0}$, $\omega^1{}_2 = 0 \Rightarrow K = 0$

Cylindrical coordinates

Cylinder curvature: $\hat{\mathbf{e}}_1 = \hat{\phi}$, $\hat{\mathbf{e}}_1 = \hat{\mathbf{z}}$, $\hat{\mathbf{n}} = \hat{\mathbf{e}}_3 = \hat{\mathbf{r}}$

$d\hat{\mathbf{n}} = d\phi\hat{\phi}$, $S = \begin{pmatrix} 1/r & 0 \\ 0 & 0 \end{pmatrix}$

$\omega^1{}_2 = -d\phi \Rightarrow d\omega^1{}_2 = 0$

$\kappa_1 = 0$, $\kappa_2 = 1/r$, Gaussian curvature vanishes

Spherical coordinates:

$d\vec{\mathbf{r}} = dr\hat{\mathbf{r}} + rd\theta\hat{\boldsymbol{\theta}} + r\sin\theta d\phi\hat{\boldsymbol{\phi}}$

$ds^2 = dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2$

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

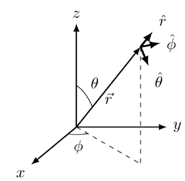
$z = r\cos\theta$

$\omega = dr \wedge rd\theta \wedge r\sin\theta d\phi$

Orthonorm: 1-forms

$\{dr, rd\theta, r\sin\theta d\phi\}$,

vecs $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\} = \{\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$



$g(d\theta, d\theta) = 1/r^2$, $g(d\phi, d\phi) = 1/(r^2\sin^2\theta)$

$u^i = r, \theta, \phi$; $h_r = 1$, $h_\theta = r$, $h_\phi = r\sin\theta$

$\hat{\mathbf{r}} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$

$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\mathbf{x} + \cos\theta\sin\phi\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}}$

$\hat{\boldsymbol{\phi}} = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}$

$d\hat{\mathbf{r}} = d\theta\hat{\boldsymbol{\theta}} + \sin\theta d\phi\hat{\boldsymbol{\phi}}$

$d\hat{\boldsymbol{\theta}} = -d\theta\hat{\mathbf{r}} + \cos\theta d\phi\hat{\boldsymbol{\phi}}$

$d\hat{\boldsymbol{\phi}} = -\sin\theta d\phi\hat{\mathbf{r}} - \cos\theta d\phi\hat{\boldsymbol{\theta}}$

$*dr = rd\theta \wedge r\sin\theta d\phi$

$*rd\theta = r\sin\theta d\phi \wedge dr$

$*rd\phi = dr \wedge rd\theta$

$dx = \sin\theta\cos\phi dr + r\cos\theta\cos\phi d\theta - r\sin\theta\sin\phi d\phi$

$dy = \sin\theta\sin\phi dr + r\cos\theta\sin\phi d\theta + r\sin\theta\cos\phi d\phi$

$dz = \cos\theta dr - r\sin\theta d\theta$

Connections: $\omega_{rr} = \omega_{\theta\theta} = \omega_{\phi\phi} = 0$

$\omega_{r\theta} = -d\theta = -\omega_{\theta r}$, $\omega_{r\phi} = -\sin\theta d\phi = -\omega_{\phi r}$,

$\omega_{\theta\phi} = -\cos\theta d\phi = -\omega_{\phi\theta}$

All curvature 2-forms $\Omega^i{}_j$ vanish in flat $\mathbb{E}^3$

Sphere curvature: $\hat{\mathbf{e}}_1 = \hat{\boldsymbol{\theta}}$, $\hat{\mathbf{e}}_2 = \hat{\boldsymbol{\phi}}$, $\hat{\mathbf{n}} = \hat{\mathbf{e}}_3 = \hat{\mathbf{r}}$

$d\hat{\mathbf{n}} = d\theta\hat{\boldsymbol{\theta}} + \sin\theta d\phi\hat{\boldsymbol{\phi}}$, $S = \begin{pmatrix} 1/r & 0 \\ 0 & 1/r \end{pmatrix}$

$\omega^1{}_2 = -\cos\theta d\theta$, $d\omega^1{}_2 = \sin\theta d\theta \wedge d\phi = \frac{1}{r^2}\sigma^1 \wedge \sigma^2$

$\kappa_1 = \kappa_2 = 1/r$, $K = \kappa_1\kappa_2 = 1/r^2$ (intrinsic)

Geodesic curvature of sphere:

Line of latitude: $\hat{\mathbf{T}} = \hat{\boldsymbol{\phi}}$, $\hat{\mathbf{N}} = \hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{\theta}}$

$\kappa_g ds = d\hat{\boldsymbol{\phi}} \cdot (-\hat{\boldsymbol{\theta}}) = cos\theta d\phi$

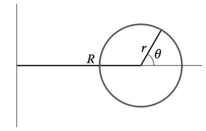
$\kappa_g = 0 \Leftrightarrow \theta = \frac{\pi}{2}$ (great circles)

Torus

$x = (R + r\cos\theta)\cos\phi$

$y = (R + r\cos\theta)\sin\phi$

$z = r\sin\theta$



$ds^2 = r^2d\theta^2 + (R + r\cos\theta)^2d\phi^2$

Orthonorm basis of 1-forms ($\omega = \sigma^\theta \wedge \sigma^\phi$):

$\sigma^\theta = rd\theta$, $\sigma^\phi = (R + r\cos\theta)d\phi$

Structure eqns: $-\omega^\theta{}_\phi \wedge \sigma^\phi = d\sigma^\theta = 0$,

$-\omega^\phi{}_\theta \wedge \sigma^\theta = d\sigma^\phi = -r\sin\theta d\theta \wedge d\phi$

$\Rightarrow \omega^\theta{}_\phi = \sin\theta d\phi$

$\Rightarrow d\omega^\theta{}_\phi = \cos\theta d\theta \wedge d\phi = \frac{\cos\theta}{r(R + r\cos\theta)}\omega$

Gaussian curvature $K = \frac{\cos\theta}{r(R + r\cos\theta)}$

Gauss-Bonnet: $\int_T K\omega = 2\pi\chi(\mathbb{T})$, $\chi(\mathbb{T}) = 0$